ADVANCED TECHNIQUES OF INTEGRATION

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Foreword. The following is a collection of advanced techniques of integration for indefinite integrals beyond which are typically found in introductory calculus courses. While we usually begin working with the general cases, it might be helpful to construct an example and model it after the general case, or use one of the exercises at the end of the section as a starting point.

Contents

1. Basic Techniques

We begin with a collection of quick explanations and exercises using standard techniques to evaluate integrals that will be used later on.

1.1. **The Decomposition Method.**

As we begin using more advanced techniques, it is important to remember fundamental properties of the integral that allow for easy simplifications. For instance, instead of using some more complicated substitution for something such as $\int (x^2+1)x \, \mathrm{d}x$, we might instead use the linearity of the integral to write

$$\int (x^2+1)x \, \mathrm{d}x = \int (x^3+x) \, \mathrm{d}x = \int x^3 \, \mathrm{d}x + \int x \, \mathrm{d}x = \frac{x^4}{4} + \frac{x^2}{2} + C.$$

This section has no exercises of its own; instead, be on the lookout for use of this simple technique in more complex integrals.

1.2. **Transforming the Integrand (Simple Substitution).**

By the chain rule for derivatives,

$$u(v(y))' = v'(y) \cdot u'(v(y)) \Rightarrow u(v(y)) = \int u'(v(y)) \cdot v'(y) \, \mathrm{d}y,$$

where $x = v(y)$. Then the rule allows us to calculate the integral of a less complicated function provided we make an appropriate transformation of the differential.

Consider, for example, $\int x^2(x^3+1)^{100} \, \mathrm{d}x$. While it is possible to expand the binomial by the binomial theorem and then use the decomposition method, it is easier to substitute $u = x^3+1$. Then the differential should be $\mathrm{d}u = 3x^2 \, \mathrm{d}x$. Hence,

$$
\begin{aligned}
\int x^2(x^3+1)^{100}\,\mathrm{d}x &= \int \frac{1}{3}u^{100}\cdot(3x^2\,\mathrm{d}x) \\
&= \frac{1}{3}\int u^{100}\,\mathrm{d}u \\
&= \frac{1}{303}u^{101}+C \\
&= \frac{1}{303}(x^3+1)^{101}+C
\end{aligned}
$$

It is important to remember that u was a variable we made up (to represent x^3+1) and that it has no meaning to an outside observer. We must always substitute back in for the original variables, and in this case express our answer in terms of x.

Exercise 1.1. Evaluate the following indefinite integrals by suitably transforming the integrand.

(1) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$

(2) $\displaystyle\int \frac{\arctan(x)}{1+x^2}\,\mathrm{d}x$

(3) $\displaystyle\int x^2\cdot\sqrt[3]{1+x^3}\,\mathrm{d}x$

(4) $\displaystyle\int x\sqrt{2-5x}\,\mathrm{d}x$

(5) $\displaystyle\int \frac{1}{(1+x)\sqrt{x}}\,\mathrm{d}x$

(6) $\displaystyle\int \sin\left(\frac{1}{x}\right)\cdot\frac{1}{x^2}\,\mathrm{d}x$

(7) $\displaystyle\int \frac{1}{(x^2+1)^{\frac{3}{2}}}\,\mathrm{d}x$

(8) $\displaystyle\int \frac{1}{x\ln x\ln(\ln(x))}\,\mathrm{d}x$

(9) $\displaystyle\int \sin(x)\sin(x+\alpha)\,\mathrm{d}x$

(10) $\displaystyle\int \frac{x^2}{\sqrt{2-x}}\,\mathrm{d}x$

(11) $\displaystyle\int \frac{1}{e^{\frac{x}{2}}+e^x}\,\mathrm{d}x$

1.3. **Integration by Parts.**

1.3.1. *Understanding the formula.* Given two differentiable functions u and v that are dependent on x, the product rule for differentiation gives

$$(uv)' = u'v + v'u.$$

Since the indefinite integral is the antiderivative, we can then write

$$
\begin{aligned}
\int (uv)'\,\mathrm{d}x &= \int (u'v+v'u)\,\mathrm{d}x \\
uv &= \int u'v\,\mathrm{d}x + \int v'u\,\mathrm{d}x \\
\int u'v\,\mathrm{d}x &= uv - \int v'u\,\mathrm{d}x
\end{aligned}
$$

Hence, when the left hand side is hard to evaluate but we can find the integral on the right hand side, we can apply this formula.

1.3.2. *Integrals of Inverses.* Applying the integration by parts formula to any differentiable function $f(x)$ gives

$$\int f(x)\,\mathrm{d}x = xf(x) - \int xf'(x)\,\mathrm{d}x.$$

In particular, if f is a monotonic continuous function, then we can write the integral of its inverse in terms of the integral of the original function f, which we denote $F(x) + C$. Substituting $f^{-1}(x)$ in the above formula we get

$$\begin{aligned}\int f^{-1}(x)\,\mathrm{d}x &= xf^{-1}(x) - \int x \cdot (f^{-1}(x))'\,\mathrm{d}x \\ &= xf^{-1}(x) - \int \big(f \circ f^{-1}(x)\big) \cdot (f^{-1}(x))'\,\mathrm{d}x \\ &= xf^{-1}(x) - F \circ f^{-1}(x) + C,\end{aligned}$$

which allows for the quick evaluation of the integral of the inverse of a multitude of functions if we know the integral of the original function.

Exercise 1.2. Evaluate the following indefinite integrals by employing the idea of integration by parts.

(1) $\int xe^{-x}\,\mathrm{d}x$

(2) $\int \sqrt{x}\ln^2(x)\,\mathrm{d}x$

(3) $\int \ln(x)\,\mathrm{d}x$

(4) $\int \arctan(x)\,\mathrm{d}x$

(5) $\int x^2 \arccos(x)\,\mathrm{d}x$

(6) $\int \sec(x)\,\mathrm{d}x$

(7) $\int \sec^3(x)\,\mathrm{d}x$

(8) $\int (\arcsin(x))^2\,\mathrm{d}x$

(9) $\int e^{\sqrt{x}}\,\mathrm{d}x$

(10) $\int \sin(\ln(x))\,\mathrm{d}x$

(11) $\int e^{ax}\cos(bx)\,\mathrm{d}x$

(12) $\int \ln(x + \sqrt{1+x^2})\,\mathrm{d}x$

1.4. **Trigonometric Substitutions.**

Exercise 1.3. Evaluate the following indefinite integrals using a suitable trigonometric substitution.

(1) $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,\mathrm{d}x$

(2) $\int \sqrt{1-x^2}\,\mathrm{d}x$

(3) $\int \sqrt{\frac{a+x}{a-x}}\,\mathrm{d}x$

(4) $\int \frac{x^2}{\sqrt{x^2-2}}\,\mathrm{d}x$

(5) $\int \frac{1}{\sqrt{(x-a)(b-x)}}\,\mathrm{d}x$

(6) $\int \sqrt{(x-a)(b-x)}\,\mathrm{d}x$

1.5. **Hyperbolic Substitutions.**

Exercise 1.4. Evaluate the following indefinite integrals by using a suitable hyperbolic substitution.

(1) $\int \sqrt{a^2+x^2}\,\mathrm{d}x$

(2) $\int \frac{x^2}{\sqrt{a^2+x^2}}\,\mathrm{d}x$

(3) $\int \frac{1}{\sqrt{(x+a)(x+b)}}\,\mathrm{d}x$

(4) $\int \sqrt{(x+a)(x+b)}\,\mathrm{d}x$

1.6. **Review Exercises.**

Exercise 1.5. Evaluate the following indefinite integrals using the methods from this section. (The integrals are not in order of difficulty.)

(1) $\int (1-x)(1-2x)(1-3x)\,\mathrm{d}x$

(2) $\int \frac{x+1}{\sqrt{x}}\,\mathrm{d}x$

(3) $\int \frac{\sin x\cos^2 x}{1+\cos^2 x}\,\mathrm{d}x$

(4) $\int \frac{e^{3x}+1}{e^x+1}\,\mathrm{d}x$

(5) $\int \sqrt[3]{1-3x}\,\mathrm{d}x$

(6) $\int \frac{2^{x+1}-5^{x-1}}{10^x}\,\mathrm{d}x$

(7) $\int \sqrt{\frac{x-a}{x+a}}\,\mathrm{d}x$

(8) $\int \frac{1}{1+\cos x}\,\mathrm{d}x$

(9) $\int \frac{\ln^2(x)}{x}\,\mathrm{d}x$

(10) $\int x\sqrt{2-5x}\,\mathrm{d}x$

(11) $\int \frac{1}{(x-1)(x+3)}\,\mathrm{d}x$

(12) $\int \frac{1}{(x^2+a^2)^{\frac{3}{2}}}\,\mathrm{d}x$

(13) $\int x^n \ln x\,\mathrm{d}x \ (n\neq -1)$

(14) $\int x^2\sqrt{a^2+x^2}\,\mathrm{d}x$

(15) $\int \frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,\mathrm{d}x$

(16) $\int (e^x-\cos x)^2\,\mathrm{d}x$

2. "Rational" Functions

2.1. **Denominator is Quadratic.**

The simplest rational function with a quadratic denominator is $\frac{1}{ax^2+bx+c}$. If we try to evaluate this integral, it is helpful to "complete the square" and write $ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)$. Then it is easier to see that for $c-\frac{b^2}{4}>0$

$$\begin{aligned}\int \frac{\mathrm{d}x}{ax^2+bx+c} &= \int \frac{\mathrm{d}x}{a\left(x+\frac{b}{2a}\right)^2+\sqrt{\left(c-\frac{b^2}{4a}\right)}^2}\\ &= \frac{1}{a}\int \frac{\mathrm{d}x}{u^2+v^2}\\ &= \frac{1}{av}\arctan\left(\frac{u}{v}\right)+C,\end{aligned}$$

where $u = x+\frac{b}{2a}$ and $v=\sqrt{\frac{c-\frac{b}{2a}}{a}}$.

In the case that $c-\frac{b^2}{4}\leq 0$, then the denominator can be factored and evaluated using partial fraction decomposition.

This technique can be applied to integrals where the situation may not be so obvious. For instance, for the integral

$$\int \frac{(x+1)}{x^2+x+1}\,\mathrm{d}x$$

one might try an immediate substitution, which would fail. First split the numerator $x+1 = \left(x+\frac{1}{2}\right)+\frac{1}{2}$ to match the denominator written as $\left(x+\frac{1}{2}\right)^2+\frac{1}{2}$. Now it is more obvious how to apply the above technique along with a substitution as

$$\begin{aligned}\int \frac{(x+1)}{x^2+x+1}\,\mathrm{d}x &= \int \frac{x+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{1}{2}}\,\mathrm{d}x + \int \frac{\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{1}{2}}\,\mathrm{d}x \\ &= \frac{1}{2}\ln(4x^2+4x+3)+\frac{1}{\sqrt{2}}\tan\left(\frac{2x+1}{\sqrt{2}}\right)+C\end{aligned}$$

2.2. **Quadratics Beneath Radicals.** For integrals of the form

$$\int \frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}}$$

we combine the ideas of completing the square with the trigonometric substitution $x = a\tan\theta$ (if $a > 0$) or $x = a\sin\theta$ (if $a < 0$).

2.3. **Ostrogradsky's Method.**

Exercise 2.1. Evaluate the following indefinite integrals by using Ostrogradsky's Method.

(1) $\displaystyle\int \frac{x}{(x-1)^2(x+1)^3}\,\mathrm{d}x$

(2) $\displaystyle\int \frac{1}{(x^3+1)^2}\,\mathrm{d}x$

(3) $\displaystyle\int \frac{1}{(x^2+1)^3}\,\mathrm{d}x$

(4) $\displaystyle\int \frac{1}{(x^4+1)^2}\,\mathrm{d}x$

(5) $\displaystyle\int \frac{x^2}{(x^2+2x+2)^2}\,\mathrm{d}x$

(6) $\displaystyle\int \frac{x^2+3x-2}{(x-1)(x^2+x+1)^2}\,\mathrm{d}x$

2.4. **Powers of Polynomials.**

Exercise 2.2. Find a recurrence relation to evaluate the integral $I_n = \displaystyle\int \frac{1}{(ax^2+bx+c)^n}\,\mathrm{d}x$, where $a \neq 0$. Use that result to evaluate $I_3 = \displaystyle\int \frac{1}{(x^2+x+1)^3}\,\mathrm{d}x$. The identity $4a(ax^2+bx+c) = (2ax+b)^2+(4ac-b^2)$ should be useful.

Exercise 2.3. Use the substitution $t = \frac{x+a}{x+b}$ to evaluate the given indefinite integrals, where m and n are integers.

(1) $\displaystyle\int \frac{1}{(x-2)^2(x+3)^2}\,\mathrm{d}x$

(2) $\displaystyle\int \frac{1}{(x+a)^m(x+b)^n}\,\mathrm{d}x$

2.5. **Taylor's Formula.**

Exercise 2.4. If P_n is some polynomial of degree n, then by use of Taylor's formula, evaluate the indefinite integral $\displaystyle\int \frac{P_n(x)}{(x-a)^{n+1}}\,\mathrm{d}x$.

2.6. **Transformation to Rational Function.** When the denominator of a fraction is not rational, we cannot apply the methods above directly to evaluate its indefinite integral. First, we make a change of variables that eliminates the irrational factors.

For instance, to evaluate

$$\int \frac{1}{1+\sqrt{x}}\,\mathrm{d}x,$$

there is no partial fraction decomposition that will help. Therefore we first take $u = \sqrt{x}$ so that

$$\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x \Rightarrow \mathrm{d}x = 2\sqrt{x}\,\mathrm{d}u = 2u\,\mathrm{d}u.$$

The original indefinite integral is hence equal to $\displaystyle\int \frac{2u}{1+u}\,\mathrm{d}u$. We can proceed as in the previous methods by dividing the numerator into the denominator.

Exercise 2.5. Evaluate the following indefinite integrals by transforming the integrand to a rational function by a suitable change of variables.

(1) $\displaystyle\int \frac{1}{x(1+2\sqrt{x}+\sqrt[3]{x})}\,\mathrm{d}x$

(2) $\displaystyle\int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}\,\mathrm{d}x$

(3) $\displaystyle\int \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}\,\mathrm{d}x$

(4) $\displaystyle\int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}}\,\mathrm{d}x$

(5) $\displaystyle\int \frac{1}{1+\sqrt{x}+\sqrt{1+x}}\,\mathrm{d}x$

(6) $\displaystyle\int \frac{1-x+x^2}{\sqrt{1+x-x^2}}\,\mathrm{d}x$

2.7. **Rational Functions with Irrational Factors.** If $P_n(x)$ is a polynomial of degree n, and if $y = \sqrt{ax^2+bx+c}$, then there exist some real number λ and $n-1$ degree polynomial $Q_{n-1}(x)$such that

$$\int \frac{P_n(x)}{y}\,\mathrm{d}x = Q_{n-1}(x)y + \lambda \int \frac{\mathrm{d}x}{y}.$$

We may verify this result by writing in the definition for y and differentiating to get that

$$\frac{P_n(x)}{\sqrt{ax^2+bx+c}} = Q_{n-2}\sqrt{ax^2+bx+c} + \lambda\frac{1}{\sqrt{ax^2+bx+c}} \Rightarrow P_n(x) = \overline{Q_n}(x) + \lambda$$

for some other nth degree polynomial, $\overline{Q_n}(x)$.

Exercise 2.6. Generalize the above formula for the indefinite integral, $\displaystyle\int \frac{P(x)}{Q(x)y}\,\mathrm{d}x$. (Hint: Consider the partial fraction decomposition of $\frac{P(x)}{Q(x)}$.)

Exercise 2.7. Using the above ideas or another method, evaluate the following indefinite integrals.

(1) $\displaystyle\int \frac{x^2}{\sqrt{1+2x-x^2}}\,\mathrm{d}x$

(2) $\displaystyle\int \frac{x^2}{\sqrt{1+x+x^2}}\,\mathrm{d}x$

(3) $\displaystyle\int \frac{a_1x^2+b_1x+c_1}{ax^2+bx+c}\,\mathrm{d}x$

(4) $\displaystyle\int \frac{x}{(x-1)^2\sqrt{1+2x-x^2}}\,\mathrm{d}x$

(5) $\displaystyle\int \frac{1}{(1-x)^2\sqrt{1-x^2}}\,\mathrm{d}x$

(6) $\displaystyle\int \frac{\sqrt{x^2+2x+2x}}{}\mathrm{d}x$

(7) $\displaystyle\int \frac{\sqrt{x^2+x+1}}{(x+1)^2}\,\mathrm{d}x$

2.8. **Fractional Linear Transformations + Euler's Substitutions.** Euler found that for integrals involving square roots of quadratics, the following substitutions are sometimes helpful.

- If $a > 0$, then set $\sqrt{ax^2+bx+c} = \pm\sqrt{ax} + z$ for some suitable choice of z.
- If $c > 0$, then set $\sqrt{ax^2+bx+c} = xz \pm \sqrt{c}$ for some suitable choice of z.
- Or sometimes take $\sqrt{a(x-x_1)(x-x_2)} = z(x-x_1)$.

Exercise 2.8. Evaluate the following indefinite integrals by using an Euler Substitution.

(1) $\displaystyle\int \frac{1}{x+\sqrt{x^2+x+1}}\,\mathrm{d}x$

(2) $\displaystyle\int \frac{1}{1+\sqrt{1-2x-x^2}}\,\mathrm{d}x$

(3) $\displaystyle\int x\sqrt{x^2-2x+2}\,\mathrm{d}x$

2.9. **Chebyshev's Theorem.** The integrals we have treated thus far have either been square roots of functions or functions raised to integral powers. The indefinite integral of many functions with other rational exponents cannot be written in terms of elementary functions. In particular,

$$\int x^m(a+bx^n)^p\,\mathrm{d}x$$

with m, n, p rationals can be reduced to the integral of a rational function only in the following three cases:

- **p is an integer:** Set $x = z^N$, where N is the least common denominator of the fractions m and n.
- **$\frac{m+1}{n}$ is an integer:** Set $a + bx^n = z^N$, where N is the denominator of the fraction p.
- **$\frac{m+1}{n} + p$ is an integer:** Set $ax^{-n} + b = z^N$, where N is the denominator of the fraction p.

This result is due to Chebyshev. We will motivate the substitutions in each of the given cases; however, the proof that these are the only cases in which there exists a clever substitution is beyond the scope of this manual.

2.9.1. *Case p is an integer.* When p is an integer, we could continue with previous techniques if the integral were transformed so that m and n were integers. Taking $z^N = x$ with N the least common denominator of m and n gives $x^m = (z^N)^m = z^{m'}$ for some integer m'. Similarly, $x^n = (z^N)^n = z^{n'}$. Also, $\mathrm{d}x = Nz^{N-1}d\,\mathrm{d}z$ and so the integral

$$\int N \cdot z^{m'+(N-1)}(a+bz^{n'})^p\,\mathrm{d}z$$

can be evaluated by a multitude of methods, including expanding by the binomial theorem and using the decomposition method.

2.9.2. *Case $\frac{m+1}{n}$ is an integer.* Since p is not necessarily an integer, we focus the substitution on the binomial term. If the binomial were raised to the power of the denominator of p, say N, then Np is an integer. So let $z^N = a + bx^n$, where $x^m = \left(\frac{z^N - a}{b}\right)^{\frac{m}{n}}$. This substitution works because $Nz^{n-1}\,\mathrm{d}z = bnx^{n-1}\,\mathrm{d}x$, so the integral becomes

$$\int \frac{\left((z^N - a)/b\right)^{\frac{m}{n}}}{\left((z^N - a)/b\right)^{\frac{n-1}{n}}} \cdot z^{N-1} \cdot z^{Np}\,\mathrm{d}z.$$

Furthermore, $\frac{m}{n} - \frac{n-1}{n}$ is an integer if $\frac{m-n+1}{n} = \frac{m+1}{n} - 1$ is an integer. Since this is that case in particular, this substitution works to make all the powers integers and we may continue as before.

2.9.3. *Case $\frac{m+1}{n} + p$ is an integer.* The verification that the substitution $ax^{-n} + b = z^N$ works is similar to the previous case but a bit more technical. It is left as an exercise to the interested reader.

Exercise 2.9. Evaluate the follow indefinite integrals using Chebyshev's theorem.

(1) $\displaystyle\int \sqrt{x^3 + x^4}\,\mathrm{d}x$

(2) $\displaystyle\int \frac{1}{\sqrt[3]{1 + x^3}}\,\mathrm{d}x$

(3) $\displaystyle\int \frac{1}{\sqrt[4]{1 + x^4}}\,\mathrm{d}x$

(4) $\displaystyle\int \sqrt[3]{3x - x^3}\,\mathrm{d}x$

3. Trig Integrals

We will assume competency with the integrals of common trigonometric expressions.

3.1. **"Rational" Trig Functions.** When confronted with an integral of the form

$$\int \frac{A\sin x + B\cos x}{a\sin x + b\cos x}\,\mathrm{d}x,$$

we need to exploit the relationship that $\frac{d}{dx}(a\sin x + b\cos x) = a\cos x - b\sin x$. We can write the numerator as a linear combination $c_1(a\sin x + b\cos x) + c_2(a\cos x - b\sin x)$ in order to use the decomposition method to evaluate the integral as

$$\int \left(c_1 + c_2\frac{u'}{u}\right)\mathrm{d}x, \text{ with } u = a\sin x + b\cos x,$$

which evaluates to $c_1 x + c_2 \ln|a\sin x + b\cos x| + C$. The constants c_1 and c_2 are determined by the equation $A\sin x + B\cos x = c_1(a\sin x + b\cos x) + c_2(a\cos x - b\sin x)$. Comparing coefficients of the sine and cosine terms gives us a system of equations that can be solved to get

$$c_1 = \frac{aA + bB}{a^2 + b^2} \qquad \text{and} \qquad c_2 = \frac{aB - bA}{a^2 + b^2}.$$

In contrast, integrals of the form

$$\int \frac{1}{a\sin x + b\cos x}\mathrm{d}x$$

do not have the numerator to allow for helpful cancellation. We know that a linear combination of trig functions is in turn a simple trig function. Our goal is to turn the denominator into an expression of the form $A\sin(x+\phi)$ for some constant phase angle ϕ. Using the angle sum formula, this is $A\cos\phi\sin x + A\sin\phi\cos x$. We can set this equal to the original denominator,

$$a\sin x + b\cos x = (A\cos\phi)\sin x + (A\sin\phi)\cos x$$

and equate the coefficients as $a = A\cos\phi$ and $b = A\sin\phi$. Then $A = \sqrt{a^2+b^2}$ and $\tan\phi = \frac{b}{a}$. So all we need to do is calculate

$$\int \frac{1}{A\sin(x+\phi)}\mathrm{d}x = \frac{1}{A}\int \csc(x+\phi)\mathrm{d}x = \frac{-1}{A}\ln|\csc(x+\phi)+\cot(x+\phi)| + C$$

and substitute in for A and ϕ.

3.2. **Hyperbolic Functions.** When dealing with integrals of hyperbolic functions, there are two general ways to proceed. We can either evaluate them with the substitution $e^x = u$ or using hyperbolic identities, such as $\cosh^2 x - \sinh^2 x = 1$.

The substitution method is more general and builds on previous sections. This is because all hyperbolic functions are rational functions of e^x, meaning that there exists some rational function $f(e^x) = h(x)$, where $h(x)$ is a hyperbolic function. In addition, the substitution maps $\mathrm{d}x$ to $\frac{1}{u}\,\mathrm{d}u$. Hence, after the susbtition, any hyperbolic function will be transformed into an expression involving polynomials of u in place of the hyperbolic functions.

An alternative to the substitution method is to evaluate hyperbolic integrals in the same manner as trignometric integrals, but with a slightly different identity set. Here is one such example:

$$\begin{aligned}\int \sinh^2(x)\mathrm{d}x &= \int \frac{\sinh(2x)-1}{2}\,\mathrm{d}x) \\ &= \frac{\sinh(2x)}{4} - \frac{x}{2} + C\end{aligned}$$

This following example is evaluated using the standard substitution.

$$\begin{aligned}\int \sinh^2(x)\cosh^2(x)\mathrm{d}x &= \int \frac{1}{u}\left(\frac{u^4}{16} - \frac{1}{16u^4} - \frac{1}{8}\right)\mathrm{d}x) \\ &= \frac{u^4}{64} + \frac{1}{64u^4} - \frac{\ln u}{8} + C \\ &= \frac{e^{4x}}{64} + \frac{e^{-4x}}{64} - \frac{x}{8} + C \\ &= \frac{\sinh(4x)}{32} - \frac{x}{8} + C\end{aligned}$$

In general, it is often hard to convert a polynomial of e^x back into hyperbolic form, which is one of the shortcomings of the substitution method. One of the strengths of this method is that it allows any rational function of hyperbolic functions to be evaluated using the substitution and Ostrogradsky's Method.

Lastly, one can convert a hyperbolic function using the relations $i\sinh(x) = \sin(ix)$ and $\cosh(x) = \cos(ix)$ and the substitution $ix = u$. However, this involves integration of complex numbers and will not be dealt with in this text.

Exercise 3.1. Evaluate the follow indefinite integrals containing hyperbolic functions.

(1) $\int \cosh^4(x)\,\mathrm{d}x$

(2) $\int \sinh^3(x)\,\mathrm{d}x$

(3) $\int \sinh(x)\cdot\sinh(2x)\cdot\sinh(3x)\,\mathrm{d}x$

(4) $\int \frac{1}{\sinh^2 x - 4\sinh x\cosh x + 9\cosh^2 x}\,\mathrm{d}x$

(5) $\int \sqrt{\tanh(x)}\,\mathrm{d}x$

(6) $\int \frac{1}{\sinh(x)+2\cosh(x)}\,\mathrm{d}x$

(7) $\int \frac{\cosh(x)}{3\sinh(x)-4\cosh(x)}\,\mathrm{d}x$

(8) $\int \sinh(ax)\cdot\cosh(bx)\,\mathrm{d}x$

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